Large deviations and stochastic resonance

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Abstract

Stochastic resonance can be seen as a noise induced periodic tuning effect, in which a small underlying deterministic periodicity is amplified by synchronized hopping between different domains of attraction of the stable fixed points of a dynamical system perturbed by Gaussian noise, thus creating a periodically forced diffusion. The trigger of stochastic resonance will be exits of diffusions of this type from domains.

Think of a potential diffusion, physically the motion of an overdamped particle in a potential landscape. In dimension 1, we may think of a smooth double well potential function $U : \mathbf{R} \to \mathbf{R}$, such that for instance U is smooth, possesses exactly two local minima at ± 1, and a unique local maximum at 0. Further $\lim_{x\to\pm\infty} U(x) = \infty$. Then the differential equation

$$dX_t^{\epsilon,x} = -U'(X_t^{\epsilon,x})dt, \quad X_0^{\epsilon,x} = x,$$

generates a dynamical system which possesses two *stable* fixed points ± 1 and one unstable fixed point 0. If the initial state $x \in]-\infty, 0[$, we have $\lim_{t\to\infty} X_t^{\epsilon,x} = -1$, whereas $\lim_{t\to\infty} X_t^{\epsilon,x} = 1$, if $x \in]0, \infty[$. We call the intervals containing the fixed points *domains of attraction*. The particle traveling in the potential landscape described by the solutions of this differential equation can therefore not exit a domain of attraction, once starting its motion inside.

This feature of the motion changes drastically, once noise is added to the system. Let $(\Omega, \mathbf{F}, P)$ with the coordinate process $W = (W_t)_{t\geq 0}$ be a one-dimensional canonical Wiener space, with the canonical filtration $(\mathbf{F}_t)_{t\geq 0}$. So for $\epsilon > 0$ we can consider the stochastic differential equation

$$dX_t^{\epsilon,x} = -U'(X_t^{\epsilon,x})dt + \sqrt{\epsilon}dW_t, \quad X_0^{\epsilon,x} = x.$$

By adding noise, however small its intensity ϵ may be, transitions between the states ± 1 become possible. We then call the states *meta-stable*, and the domains of attraction lose their original meaning. However, in the *small noise limit*, i.e. as $\epsilon \to 0$, the particle starting in one of them will typically, i.e. with high probability, leave this domain in times depending on ϵ and given by the asymptotic quantities

$$\exp(\frac{2(U(0) - U(-1))}{\epsilon}) \quad \text{if} \quad x \in]-\infty, 0[,$$

and

$$\exp(\frac{2(U(0) - U(1))}{\epsilon}) \quad \text{if} \quad x \in]0, \infty[.$$

To show this, Freidlin and Wentzell applied the theory of *large deviations* for diffusion processes perturbed by small Gaussian noise. We shall develop the essentials of this theory, and then present the law of asymptotic exit times, noted before in papers by chemists and physicists occupied with phenomena of reaction- diffusion. In this context, large deviations concern the asymptotic behavior of the laws $\mu_\epsilon = P \circ X^{\epsilon,x}$, as $\epsilon \to 0$. We shall start with an approach of the large deviations principle (LDP) for the Brownian motion W, in which the theorem of Schilder is derived from a Fourier series decomposition of W. The LDP is transferred via continuity to diffusions.